BRiTE: Bootstrapping Reinforced Thinking Process to Enhance Language Model Reasoning

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Reasoning as a Graphical Model

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Q: What is reasoning in large language models?
A: Okay, so I need to figure out what reasoning
   is in large language models (LLMs) is. Let
   me start by breaking down the question. The
   user is asking about reasoning ...
Reasoning in large language models (LLMs) refers
to their ability to generate responses that mimic
structuredlogical thought processes to solve
problems or answer questions.
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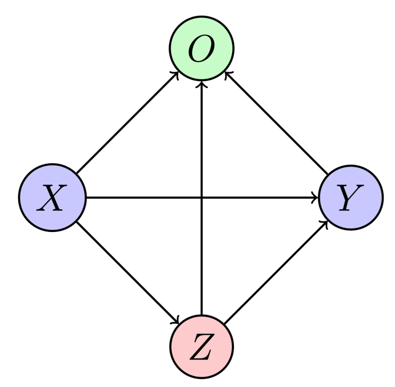


$$\mathbb{P}(y, o \mid x, \theta) = \mathbb{P}(z \mid x, \theta) \cdot \mathbb{P}(y \mid x, z, \theta) \cdot \mathbb{P}(o \mid x, z, \theta)$$

Bootstrap Reinforced Thinking Process

$$\mathscr{L}(\theta) = \log \sum_{(z,y,o)\in\mathscr{Z}\times\mathscr{Y}\times\mathscr{O}} \mathbb{P}(z, y, o \mid x, \theta)$$
$$= \max_{\mathbb{Q}\in\Delta} \Big\{ \underbrace{\sum_{(z,y,o)} \log \mathbb{P}(z, y, o \mid x, \theta)\mathbb{Q}(z, y, o \mid x, \psi) - \sum_{(z,y,o)} \log \mathbb{Q}(z, y, o \mid x, \psi)\mathbb{Q}(z, y, o \mid x, \psi)}_{:=\mathscr{L}_\psi(\theta)} \Big\}$$

Maximize $\mathscr{L}(\theta)$ (difficult) $\Longrightarrow$ Maximize evidence lower bound $\mathscr{L}_\psi(\theta)$ (easy)

BRiTE — An EM-type Algorithm

E (Thought proposer):

$$\mathbb{Q}(z, y, o \mid x, \psi_{t+1}) \leftarrow \text{argmax}_{\mathbb{Q}} \mathscr{L}_\psi(\theta_t) = \frac{\mathbb{P}(z, y, o \mid x, \theta_t)}{\sum_{(z,y,o)} \mathbb{P}(z, y, o \mid x, \theta_t)}$$

M:

$$\theta_{t+1} = \text{argmax}_\theta \mathscr{L}_{\psi_{t+1}}(\theta) = \text{argmax}_\theta \Big\{ \sum_{z,y,o\in\mathscr{Z}\times\mathscr{Y}\times\mathscr{O}} \log \mathbb{P}(z, y \mid x, \theta) \cdot \mathbb{Q}(z, y, o \mid x, \psi_{t+1}) \Big\}$$

Assumptions: $f_\theta \in \mathscr{H}$ for a certain RKHS; $\mathbb{P}(z, y \mid x, \theta) \propto \exp(f_\theta(x, z, y))$

Theorem: convergence to optima

$$\min_{1\le t\le T} \Big\{ \log \frac{\mathbb{P}(x \in \mathscr{X}, y \in \mathscr{Y}, o \in \mathscr{O} \mid x, \theta^*)}{\mathbb{P}(x \in \mathscr{X}, y \in \mathscr{Y}, o \in \mathscr{O} \mid x, \theta_t)} \Big\} \le \frac{\mathbb{D}_{\mathrm{KL}}(\mathbb{P}(\cdot \mid x, \theta_1) \| \mathbb{P}(\cdot \mid x, \theta^*))}{T}$$

Concrete Examples of BRiTE

Scope:
- $o \in \{0,1\}$, $\mathscr{O} = \{1\}$
- $\mathscr{Y}$ is the response space
- $\mathscr{Z}$ is the latent space

- $\mathbb{P}(o = 1 \mid x, z, y) := \exp(R(x, z, y)/\beta)$
- $\mathbb{P}(z, y, o = 1 \mid x, \theta) = \mathbb{P}(z, y \mid x, \theta)\mathbb{P}(o = 1 \mid x, z, y)$
- $\mathbb{Q}(z, y \mid x, \psi) := \mathbb{Q}(z, y, o = 1 \mid x, \psi)$

Example (PPO)

$$\mathscr{L}_\psi(\theta) = \sum_{(z,y)} \log \mathbb{P}(z, y, o = 1 \mid x, \theta)\mathbb{Q}(z, y \mid x, \psi) - \sum_{(z,y)} \log \mathbb{Q}(z, y \mid x, \psi)\mathbb{Q}(z, y \mid x, \psi)$$
$$= \mathbb{E}_{(z,y)\sim\mathbb{Q}}\left[R(x, z, y)/\beta - \log \frac{\mathbb{Q}(z, y \mid x, \psi)}{\mathbb{P}(z, y \mid x, \theta)}\right]$$

Scope:
- $o \in \{0,1\}$, $\mathscr{O} = \{1\}$
- $\mathscr{Y}$ is the response space
- $\mathscr{Z}$ is the latent space

- $\mathbb{P}(o = 1 \mid x, z, y) := \mathbb{I}(y \text{ is correct for } x)$ or $\exp(R(x, y)/\beta)$
- $\mathbb{P}(z, y, o = 1 \mid x, \theta) = \mathbb{P}(z, y \mid x, \theta)\mathbb{P}(o = 1 \mid x, z, y)$
- $\mathbb{Q}(z, y \mid x, \psi) := \mathbb{Q}(z, y, o = 1 \mid x, \psi)$

$$\max_{\mathbb{P}} \Big\{ \mathbb{E}_{(z,y)\sim\mathbb{P}(\cdot,\cdot \mid x,\theta_t)}\big[\log \mathbb{P}(z, y \mid x, \theta) \cdot \mathbb{I}(y \text{ is correct for } x)\big] \Big\}$$

$$\max_{\mathbb{P}} \Big\{ \mathbb{E}_{(z,y)\sim\mathbb{P}(\cdot,\cdot \mid x,\theta_t)}\big[\log \mathbb{P}(z, y \mid x, \theta) \cdot \exp(R(x, y)/\beta)\big] \Big\}$$

If $\mathscr{Z} = \emptyset$, then it recovers STaR and Reject Sampling Finetuning or RestEM

Learning Intractable Posterior via RL

$$\mathbb{Q}(z, y, o \mid x, \psi) \leftarrow \text{argmax}_{\mathbb{Q}} \mathscr{L}_\psi(\theta) = \frac{\mathbb{P}(z, y, o \mid x, \theta)}{\sum_{(z,y,o)} \mathbb{P}(z, y, o \mid x, \theta)}$$

Intractable

Lemma: the optimal policy for an entropy-regularized token-level MDP

$$\pi^\star(a_h \cup \{(s_i, a_i)\}_{i=h+1}^H \mid s_h) \propto \exp\left(\frac{1}{\beta} \sum_{i=h}^H r(s_i, a_i)\right)$$

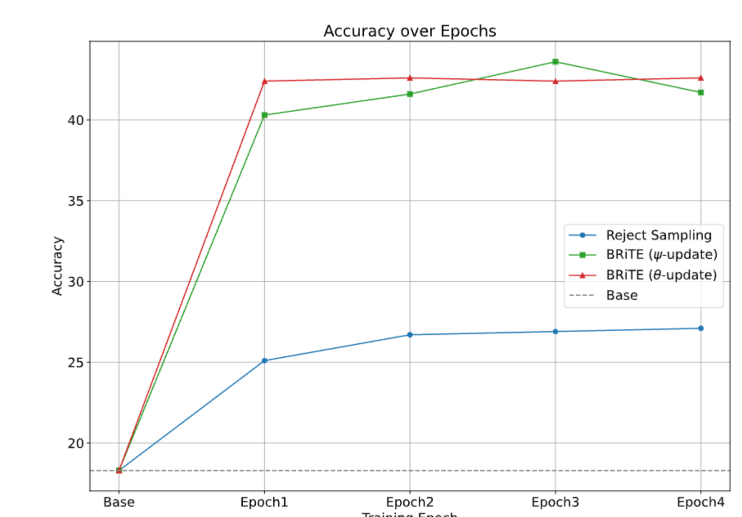
Set $\frac{1}{\beta} \sum_{i=0}^H r(s_i, a_i) = \log \mathbb{P}(z, y, o \mid x, \theta)$! Then $\pi^\star(s_H \mid s_0) = \mathbb{Q}(z, y, o \mid x, \psi)$

Experiments

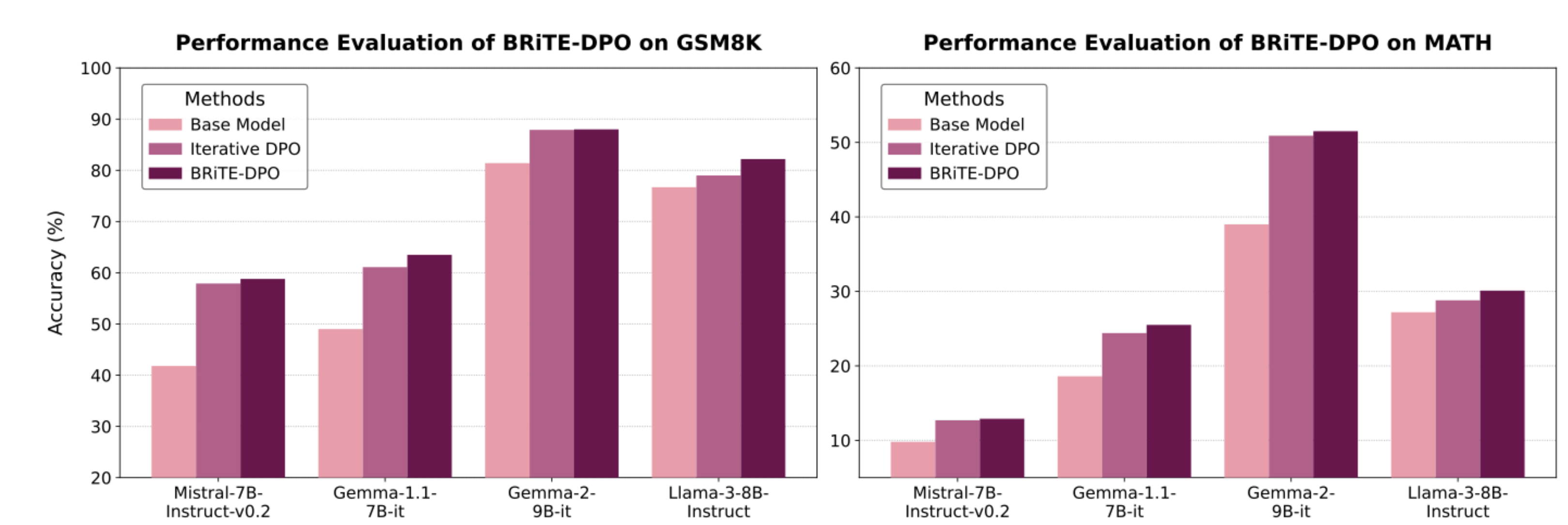
1. BRiTE Significantly Improves Existing Rejection Sampling Algorithms.
2. BRiTE ≥ SFT with Human-Annotated Thinking Process.

Method	MATH500	Minerva Math	OlympiadBench	AIME24	AMC23	GPQA Diamond
—	44.1	12.9	16.1	0.9	10.1	25.9
RS	54.3	21.0	23.1	5.6	31.6	26.9
BRiTE (ψ-update)	79.1	35.0	35.7	14.3	57.7	28.5
BRiTE (θ-update)	76.9	40.6	37.0	14.4	57.1	29.8
BRiTE-iter-2 (ψ-update)	**80.6**	**41.3**	37.3	14.3	**57.9**	29.9
BRiTE-iter-2 (θ-update)	78.2	39.8	**37.9**	**15.3**	56.4	**30.1**

3. BRiTE Generates High Quality Trajectories for Distillation.



4. BRiTE Enhances the Reasoning and Coding Capacity in RLHF Stage.



Algorithm	HumanEval		BCB (Instruct)	
	Basic (%)	Plus (%)	Hard (%)	Full (%)
—	78.0	70.7	10.1	35.5
SFT	78.0	67.7	11.5	**37.2**
RS	79.3	**73.2**	11.5	35.6
BRiTE	**81.7**	72.6	**15.5**	36.3